Extremal eigenvalues of sparse random graphs

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thesis defence

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Spectral gap of sparse random digraphs
Let $G = (V, E)$ be any finite graph (directed or undirected)

- (out)-degree of vertex $x$:

$$\deg_G(x) = |\{y \in V : (x, y) \in E\}|$$

- Transition matrix of the simple random walk on $G$:

$$P_{x,y} = \frac{1}{\deg_G(x)} \mathbf{1}\{(x, y) \in E\}$$

- We order the eigenvalues of $P$ by decreasing modulus:

$$1 = |\lambda_1(P)| \geq \cdots \geq |\lambda_n(P)|$$
Introduction: Ramanujan graphs, Ramanujan digraphs

\[ G_n, d = \text{set of undirected } d\text{-regular graphs on } n \text{ vertices (} n \text{ even)} \]

\[ G_n = \text{uniform random variable on } G_n, d \]

Theorem (Alon-Friedman)

Then with high probability as \( n \to \infty \), we have

\[ |\lambda_2(P_n)| \rightarrow \frac{2}{\sqrt{d}} - \frac{1}{d}. \]

Question

What happens with \( \vec{G_n}, d \), the set of uniform \( d\)-regular directed graphs?

Conjecture [Parzanchevski]:

\[ |\lambda_2(P_n)| \rightarrow 1 \frac{\sqrt{d}}{d}. \]
Introduction: Ramanujan graphs, Ramanujan digraphs

$\mathcal{G}_{n,d}$ = set of undirected $d$-regular graphs on $n$ vertices ($nd$ even)
$G_n$ = uniform random variable on $\mathcal{G}_{n,d}$
$P_n$ = transition matrix of the simple random walk on $G_n$

**Theorem (Alon-Friedman)**

*Then with high probability as $n \to \infty$, we have*

$$|\lambda_2(P_n)| \xrightarrow{n \to \infty} \frac{2\sqrt{d - 1}}{d}.$$
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Question

What happens with $\bar{\mathcal{G}}_{n,d}$, the set of uniform $d$-regular directed graphs?

Conjecture [Parzanchevski]: $|\lambda_2(P_n)| \to \frac{1}{\sqrt{d}}$. 
The model: directed configurations

\[ \mathbf{d} = (d + 1, d - 1, \ldots, d + n, d - n) \]

sequence of integers greater than 2

\[ M(d) = \text{set of directed multigraphs on } n \text{ vertices, such that for every vertex } i, \]
\[ \text{indegree}(i) = d - i \quad \text{outdegree}(i) = d + i \]

We must have
\[ d + 1 + \cdots + d + n = d - 1 + \cdots + d - n = m \]
if we want \( M(d) \neq \emptyset \).

\[ G = \text{uniform random variable on } M(d) \]
\[ P = \text{transition matrix of the SRW on } G \]

Difficulties

- \( P \) is not normal or 'almost' normal.
- The eigenvalues of \( P \) are complex + no min-max characterizations.
- The stationary measure \( \pi \) is unknown when \( d \) is not constant.
The model: directed configurations

- $\mathbf{d} = (d_1^+, d_1^-, \ldots, d_n^+, d_n^-)$ sequence of integers greater than 2
- $\mathcal{M}(\mathbf{d}) =$ set of directed multigraphs on $n$ vertices, such that for every vertex $i$,
  \[ \text{indegree}(i) = d_i^- \quad \text{outdegree}(i) = d_i^+ \]

- We must have $d_1^+ + \cdots + d_n^+ = d_1^- + \cdots + d_n^- := m$ if we want $\mathcal{M}(\mathbf{d}) \neq \emptyset$.
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Result: spectral gap for directed configurations

Let $d^{(n)} = (d_1^+, d_1^-, \ldots, d_n^+, d_n^-)$ be integer sequences.

- $\sum d_i^+ = \sum d_i^- := m$ is the number of directed edges.
- There is an integer $\Delta \geq 2$ such that for every $n$,

$$2 \leq \delta = \min d^{(n)} \leq \max d^{(n)} \leq \Delta.$$

Theorem (C., 2017)

Let $G_n$ be a uniform random variable on $\mathcal{M}(d^{(n)})$, with transition matrix $P_n$. For any $\varepsilon > 0$, with high probability as $n$ goes to infinity,

$$|\lambda_2(P_n)| \leq \max \left( \frac{1}{\delta}, \sqrt{\frac{1}{m} \sum_{i=1}^{n} \frac{d_i^-}{d_i^+}} \right) + \varepsilon.$$
Result: spectral gap for directed configurations

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\]

Corollary: in a random directed \( d \)-regular (multi)graph, \( |\lambda_2(P_n)| \leq 1/\sqrt{d} + \varepsilon \) whp.
The degree sequence $d^{(n)}$ is chosen so that

$$\max\{\delta^{-1}, \rho\} = \rho$$

Circle in red = $\rho$, circle in green = $\delta^{-1}$
Spectrum of $P_n$
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Circle in red = $\rho$, circle in green = $\delta^{-1}$
Extended states at zero in sparse Erdős-Rényi graphs
Erdős-Rényi graphs

$G_n \sim \text{Erdős-Rényi } (n,p)$

- $G_n$ is an undirected loopless graph on $n$ vertices
- Each edge $(x,y)$ appears independently with probability $p$.

$A_n = \textbf{adjacency} \text{ matrix of } G_n \text{ (Hermitian)}$
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Empirical spectral distribution of the spectrum of $A$:

$$
\mu_n := \frac{1}{n} \sum_{i=1}^{n} \delta_{\lambda_i(A)}(A)
$$
Eigenvalues of Erdős-Rényi graphs, dense case

Spectrum of a realization of an undirected Erdős-Rényi graph

\[ n = 10000, p = \frac{1}{2} \]

This is Wigner’s semicircle distribution (rescaled).
Closed form, absolutely continuous, bounded support, bounded density...
Pictures: eigenvalues of Erdős-Rényi graphs, sparse case
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Spectrum of a realization of an undirected Erdős-Rényi graph

\[ n = 10000, p = c/n \]
Almost surely, $\mu_n$ converges weakly to some deterministic probability measure $\mu_c$. 

[Zakharevich, 06] [Bordenave, Lelarge, 07] [Abért, Thom, Virág, 16]...
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What are the properties of $\mu_c$?
What happens around zero? The [Bauer, Golinelli, 2000] conjecture
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$c = 2$

$c = 2, 6$

$c = 2, 8$

$c = 3$

(log scale for the $x$-axis)
**Definition**
We say that a measure $\mu$ has no extended states at $E$ if

$$\lim_{\varepsilon \to 0^+} \frac{\mu([E - \varepsilon, E + \varepsilon]) - \mu\{E\}}{2\varepsilon} = 0.$$
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$$\lim_{\varepsilon \to 0^+} \frac{\mu([E - \varepsilon, E + \varepsilon]) - \mu(\{E\})}{2\varepsilon} = 0.$$  

Theorem (C, Salez, 2018)

- If $c < e$ then $\mu_c$ has no extended states at zero.
- If $c > e$ then $\mu_c$ has extended states at zero.

+ easy generalization [C, 19]: $\mu_{\text{ske}}$ has no extended states at zero.
Hints for the explanation of a transition at $e \approx 2.718$.

Karp and Sipser analysis (1981) of the leaf-removal algorithm applied to $G_n$:

- Leaf-removal preserves the dimension of the kernel of $A_n$.
- It yields a 2-core $C_n$ and $J_n$ isolated vertices.
- $|C_n| = o(n)$ if and only if $c \leq e$.

Value of the atom at zero [BLS 2015]:

$$\mu_c(\{0\}) = \lim_{n \to \infty} \frac{\dim \ker(A_n)}{n} = e^{-cx} + cxe^{-cx} + x - 1$$

where $x$ is the smallest solution in $(0, 1)$ of $x = e^{-ce^{-cx}}$. 
Extremal eigenvalues of weighted directed Erdős-Rényi graphs
Motivation: the problem of matrix reconstruction

\[
P = \begin{pmatrix}
1 & 7 & 6 & 1/\pi & 1.5 & e^2 \\
7 & 0 & 2^{100} & 1 & 3 & 1/7 \\
6 & 2^{100} & \sqrt{5} & 0 & 1 & \pi \\
1/\pi & 1 & 0 & 8 & -1 & 2019 \\
1.5 & 3 & 1 & -1 & 1.5 & \pi^e \\
e^2 & 1/7 & \pi & 2019 & \pi^e & \zeta(5)
\end{pmatrix}
\]
Motivation: the problem of matrix reconstruction

\[ P' = \begin{pmatrix} 7 & 7 & e^2 \\ 7 & \sqrt{5} & 0 \\ 1/7 & -1 & \pi^e \end{pmatrix} \]

Can you recover \( P \)?
Can you recover some parts of \( P \) (singular values, singular vectors)?

**General, informal answer**
If \( P \) is not too complex (low rank, delocalized entries) and if at least \( n \log n \) entries are revealed, one can recover \( P \).

[Candès-Tao 09, Candès-Recht 10, Keshavan-Montanari-Oh 09, Chatterjee 15, ...]
The mathematical model

**Hypotheses** on $P$:

1. $P$ is Hermitian with positive eigenvalues:

   $$P = \sum_{i=1}^{r} \mu_i \phi_i \phi_i^* \quad \text{with} \quad 0 < \mu_r \leq \cdots \leq \mu_1.$$

2. Low rank: $r = O(\text{polylog}(n))$.

3. Delocalized eigenvectors: $|\phi_i|_\infty = O\left(\frac{1}{\sqrt{n}}\right)$.

Each entry $P_{x,y}$ is observed independently with probability $d/n$ (we restrict to $d > 1$).

**Matrix of observations**:

$$A_{x,y} = \begin{cases} \frac{n}{d} P_{x,y} & \text{if entry } (x,y) \text{ is observed.} \\ 0 & \text{if not.} \end{cases}$$

The rescaling is chosen so that $\mathbf{E}[A] = P$. 
Define $Q_{x,y} = nP_{x,y}^2$ and $\rho = \|Q\|$. 

$$\vartheta = \sqrt{\frac{\rho}{d}} \quad \vartheta_0 = \frac{\max_{x,y} |P_{x,y}|}{d}$$

$r_0 \in \{0, \ldots, n\} = \text{number of eigenvalues of } P \text{ above the threshold } T = \max\{\vartheta, \vartheta_0\}$:

$$\mu_r \leq \cdots \leq \mu_{r_0+1} \leq \max\{\vartheta, \vartheta_0\} < \mu_{r_0} \leq \cdots \leq \mu_1$$

**Theorem (Bordenave, C., Nadakuditi, in preparation)**

*With high probability as $n \to \infty$, the $r_0$ eigenvalues of $A$ with greatest modulus are converging towards $\mu_1, \ldots, \mu_{r_0}$. All the other eigenvalues of $A$ are smaller than $\max\{\vartheta, \vartheta_0\} + o(1)$.*

**Corollary:**

$A = \text{adjacency matrix of a directed Erdős-Rényi graph with } p = d/n$. Then whp,

$$\lambda_1(A) \to d \quad \text{and} \quad \max_{k>1} |\lambda_k(A)| \leq \sqrt{d} + o(1).$$
Illustrations: spectrum of $A$ for a fixed $P$ and different $d$

$$P = \varphi_1 \varphi_1^* + 2 \varphi_2 \varphi_2^* + 3 \varphi_3 \varphi_3^*$$ with the $\varphi_i$ delocalized

For this $P$ we always have $\max\{\vartheta, \vartheta_0\} = \vartheta = \sqrt{\rho/d}$ for $d > 1$.

$d = 5$ which yields $\vartheta \approx 2.44$: in this case $\mu_1 < \mu_2 < T < \mu_3$. 
Illustrations: spectrum of $A$ for a fixed $P$ and different $d$

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For this $P$ we always have $\max\{\vartheta, \vartheta_0\} = \vartheta = \sqrt{\rho/d}$ for $d > 1$.

$d = 15$ which yields $\vartheta \approx 1.30$: in this case $\mu_1 < T < \mu_2 < \mu_3$. 
Results: recovering eigenvectors in the rank-one case

We suppose $P = \varphi \varphi^*$ (rank-one case).

Theorem (phase transition for rank-one matrices)

With high probability, the following holds:

1. If $d < n |\varphi|^4_4$ then $\max_{k \in [n]} |\lambda_k(A)| \leq \sqrt{\frac{n |\varphi|^4_4}{d}} + o(1)$.

2. If $d > n |\varphi|^4_4$:

   $\lambda_1(A) \to 1$ and $\max_{k > 1} |\lambda_k(A)| \leq \sqrt{\frac{n |\varphi|^4_4}{d}} + o(1)$.

Moreover, if $\psi$ is the normalized eigenvector of $A$ associated with $\lambda_1$, then

$$|\langle \psi, \varphi \rangle| \sim \sqrt{1 - \frac{n |\varphi|^4_4}{d}}.$$  

(Remark: $n |\varphi|^4_4 \ll 1$ because of localization of $\varphi$)

⇒ ‘sparse non-symmetric’ version of the BBP transition.
$P = \varphi \varphi^*$; we observe $A$ and want to recover $\varphi$.

Three PCA-like estimators:

1. Compute the SVD $A = \sum \sigma_i \zeta_i \xi_i^*$. Use the estimator
   \[ \hat{\varphi}_{\text{svd}} = \zeta_1. \]

2. Use the fact that $P$ is known to be Hermitian! Set $A_{\text{sym}} := (A + A^*)/2$ and write the spectral decomposition $A_{\text{sym}} = \sum \gamma_i \chi_i \chi_i^*$. Use the estimator
   \[ \hat{\varphi}_{\text{sym}_eig} := \chi_1. \]

3. Simply use the eigenvectors of $A$:
   \[ A \psi = \lambda_1 \psi \quad \text{with} \quad \lambda_1 \sim 1 \]
   and use
   \[ \hat{\varphi}_{\text{asym}_eig} = \psi_1. \]

Quality measurements of the estimators $|\langle \varphi, \hat{\varphi}_* \rangle|$ and $|\varphi - \hat{\varphi}_*|_\infty$. 
Numerics (Raj Rao N)
Proof method: ingredients

1. ‘pseudo-eigenvectors’ of $A$:

$$u_i = \frac{A^\ell \phi_i}{\mu_i^\ell} \quad v_i = \frac{(A^*)^\ell \phi_i}{\mu_i^\ell}$$

2. Diagonalizable proxy for $A$:

$$S = \sum_{i=1}^{r_0} \mu_i^\ell u_i v_i^* = UD^\ell V^*$$

3. High-trace method:

$$||A^\ell - S|| \lesssim \left[ \max\{\vartheta, \vartheta_0\} \right]^\ell = o(||S||). \quad (1)$$

4. Bauer-Fike-style arguments
Future research directions

Spectral gaps:
- Simple configuration models or non-regular sparse graphs: using the non-backtracking matrix and/or the universal cover of the graph
- Alon-Boppana bounds for non-normal operators?
- Higher-dimensional simplicial complexes?

Asymptotic spectral properties of sparse graphs:
- More on the continuous part of $\mu$?
- Monotonicity of the total mass of atoms in $c$?

Matrix reconstruction from $O(n)$ entries:
- Ongoing extensions of the results in Section 4: non-square problems $P$, using non-backtracking matrices. Extension to tensors.
- MMSE approach to the problem: fundamental limits VS our threshold
- Theory explaining why non-symmetry can perform well... even in symmetric problems.
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