Hyperuniformity

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Abstract

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Let Φ be a stationary point process on $\mathbb{R}^d$. We preliminary say that Φ is hyperuniform if it exhibits anomalously small density fluctuations:

$$\lim_{R \to \infty} \frac{\text{Var}(\Phi \cap B(0,R))}{|B(0,R)|} = 0. \quad (0)$$

where $|A|$ denotes the Lebesgue measure of the measurable set $A$ and $B(0,r)$ is the euclidean ball of radius $r$ around zero. Definition (0) is in contrast with the Poisson Point Process, where the random variable $\Phi \cap B(0,R)$ has a $\text{Poi}(|B(0,R)|)$ distribution, hence the limit in the LHS is equal to its intensity $\lambda > 0$. Hyperuniform point processes can be locally disordered, but their large-scale structure shows a better organization.

**A very short history**

Processes with reduced density fluctuations have been known for long: see for instance the Martin and Yalcin paper from 1980 on Coulomb gases, [24], or Lebowitz’ paper [23]. Several papers in the 2000s ([14, 6]) identified hyperuniformity in several physical or statistical models.

However, the seminal paper [29] by Stillinger and Torquato (2003) seems to be the starting point of the field. Since then, Torquato and his team published quite a number of various paper exploring hyperuniformity and its properties; many of them will be quoted in this text, but we refer to the survey [28] from 2018, gathering the most significant physics papers in the field. Hyperuniform systems are now popular in the materials science community and also in the physics of particle systems.

Only recently (say, 2016) did the mathematics community become seriously interested in hyperuniformity *per se*, when the pioneering papers of Ghosh and Lebowitz ([9, 12]) and their short survey ([11]) opened the path to further investigations. A key aspect of their work is the link between hyperuniformity and rigidity, a much more studied phenomenon. Recently, Chatterjee proved hyperuniformity for hierarchical Coulomb gas ([4]), and hyperuniformity has been studied in matchings of point processes by Klatt, Last and Yogeshwaran ([20]).

**This survey**

The first section gathers some technical preliminaries on Palm measures, correlations, Fourier analysis, etc. The following two sections only describe the main tools for studying hyperuniformity: Section 2 proves the elementary formulas for the variance of linear statistics of point processes and in Section 3, we discuss the basic definition of hyperuniformity (especially with regards to window shape dependence) and immediate consequences such as the quadratic formula. Then, Section 6 explores the links between rigidity phenomena and hyperuniformity. We prove that hyperuniformity and suitable decay of correlations imply rigidity in $d = 1, 2$ and we prove the extreme rigidity properties of stealthy hyperuniform processes (no bounded holes, anticoncentration, maximal rigidity), following Ghosh-Lebowitz.
1 Preliminaries

1.1 Configurations

In this survey, we will consider non-negative integer-valued Radon measures on $\mathbb{R}^d$, which are simple in the sense that the mass given to any singleton is either 0 or 1. Such a measure is called a configuration and can be identified with the discrete set of its atoms — we will do this identification whenever we want. The set of configurations on $\mathbb{R}^d$ is noted $\text{Conf}(\mathbb{R}^d)$. This space is equipped with the sigma-algebra generated by the counting functions $\varphi \in \text{Conf}(\mathbb{R}^d) \to |\varphi \cap A|$, for any Borel set $A$.

A random point process is simply a random variable on this set. An underlying probability space $(\Omega, \mathcal{F}, P)$ will be fixed and will play no role in the sequel, although the bold-font letters $E, P$ will always refer to this underlying expectation and probability operators.

Capital greek letters such as $\Phi, \Psi$ will be my favorite notation for such random variables, the small-case letters like $\varphi$ being reserved to specific elements of $\text{Conf}(\mathbb{R}^d)$. If $x$ is a point in $\mathbb{R}^d$, we will note $\varphi + x = \{y + x, y \in \varphi\}$ the set $\varphi$ shifted by $x$.

1.2 Palm technology

We recall that a point process $\Phi$ is stationary if for any fixed $x \in \mathbb{R}^d$, the shifted process $\Phi - x$ has the same distribution as $\Phi$. If this is the case, the intensity measure is a constant $\lambda$ times the Lebesgue measure. In all this survey, $\lambda$ will always denote this intensity and will always be supposed to be nonzero.

Let $\Phi$ be a random stationary PP. The Palm distribution of $\Phi$ is the probability measure $P^\circ$ defined through the formula

$$P^\circ(A) = \frac{1}{|B(0,1)|} E \left[ \int_{B(0,1)} 1_{\Phi - x \in A} dx \right].$$

(1.1)

Indeed, one can replace $B(0,1)$ by any other measurable set $B$ with positive Lebesgue measure. A simple extension procedure shows that for any measurable mapping $f$, one has

$$E^\circ[f(\Phi)] = \frac{1}{\lambda|B(0,1)|} E \left[ \int_{B(0,1)} f(\Phi - x) dx \right].$$

(1.2)

For excellent textbook-treatments of properties of stationary point processes we refer to [5, 21, 2]. We will not really use the properties of Palm distributions, except the elementary Campbell formula, which can be formulated in its simplest form as follows: if $f : \text{Conf}(\mathbb{R}^d) \times \mathbb{R}^d$ is a nonnegative measurable function, then

$$E \left[ \sum_{x \in \Phi} f(x, \Phi - x) \right] = \lambda \int_{\mathbb{R}^d} E^\circ[f(x, \Phi)] dx.$$

(1.3)

1.3 Correlations

For every integer $k$, the formula

$$f \mapsto E \left[ \sum_{x_1, \ldots, x_k \in \Phi} f(x_1, \ldots, x_k) \right]$$

(1.4)

defines a unique measure $\mu^{(k)}$ on $\mathbb{R}^d$,$^k$, which is called the $k$-point correlation measure. In the definition, the sum is over all $k$-tuples of points belonging to $\Phi$, such that all of them are distincts. It is clear that $\mu^{(1)}$ is simply the intensity measure of the process.
1.4 Covariance between linear functionals

Let $f, g$ be Schwartz functions$^1$. It is easily seen that

$$\text{Cov} \left( \sum_{x \in \Phi} f(x), \sum_{x \in \Phi} g(x) \right) = \int f(x)g(x)\mu^{(1)}(dx) + \int\int f(x)g(y)\mu^{(2)}(dx, dy) - \int\int f(x)f(y)\mu^{(1)}(dx)\mu^{(1)}(dy). \quad (1.5)$$

This formula easily extends to functions $f, g$ which are compactly supported. Note that the first term can actually be written as an integral of $(x, y) \mapsto f(x)g(y)$ with respect to the ‘disintegrated measure’ $\delta_x(dy) \otimes \mu^{(1)}(dx)$. We will now define the ‘full covariance measure’:

$$\mu_{\text{cov}} = \delta_x \otimes \mu^{(1)} + \mu^{(2)} - \mu^{(1)} \otimes \mu^{(1)}.$$

(1.6)

The first term is the contribution of each point, the second one captures the covariance between points and the third one is the density term.

**Remark 1.1.** It is not a minor fact that this object $\mu_{\text{cov}}$ is naturally defined as a tempered measure, and not as a signed measure. By ‘tempered measure’, I mean a Borel measure which integrates all the Schwartz functions, ie which can be seen as a tempered distribution$^2$. Indeed, in most cases, the measures $\mu^{(2)}$ and $\mu^{(1)}$ are both infinite and it is not trivial at all to define the meaning of $\mu^{(2)}(A) - \mu^{(1)}(A)^2$ when $A$ has infinite measure. I would like to avoid doing assumptions for the moment, so we will stick to the tempered distributions setting, for the moment.

2 Variance formulas

2.1 The direct-space variance formula

The following theorem introduces the most fundamental object, the covariance measure. We will often need to use the ‘tilted convolution’ operator, which is defined as

$$f \odot g(x) = \int f(y)g(x + y)dy. \quad (2.1)$$

**Theorem 2.1** (covariance measure). Let $\Phi$ be a stationary point process with intensity $\lambda$ on $\mathbb{R}^d$. Let $f, g \in \mathcal{S}(\mathbb{R}^d)$ be two Schwartz functions. Then, there is a tempered measure $\mu_{\circ}$ on $\mathbb{R}^d$ such that

$$\text{Cov} \left( \sum_{x \in \Phi} f(x), \sum_{x \in \Phi} g(x) \right) = \int f \odot g(x)\mathcal{C}(dx) \quad (2.2)$$

where $\mathcal{C}$ is the tempered measure on $\mathbb{R}^d$ called covariance measure, and defined by

$$\lambda\delta_0 + \lambda\mu_{\circ} - \lambda^2. \quad (2.3)$$

This theorem is exactly a disintegration formula for the full covariance measure: the right-hand-side can be rewritten as $\int f(u)g(v + u)\mathcal{C}(du)dv$, which exactly says that $\mu_{\text{cov}}$ defined earlier in (1.6) satisfies $\mu_{\text{cov}}(du, dv) = \mathcal{C}(du - dv)dv$. It can also be seen as a formulation of the existence theorem of Palm distributions$^3$.

One will easily extend formula (2.2) to functions such as $1_A$ where $A$ is a bounded measurable set. The measure $\mu_{\circ}$ mentioned in the theorem is nothing more than the ‘reduced Palm density’, ie

$$\mu_{\circ}(A) = \mathbb{E}^\circ \left[ \sum_{x \in \Phi \setminus \{0\}} 1_{x \in A} \right]. \quad (2.4)$$

---

$^1$I refer to Appendix A.1 for reminders on basic analysis, Schwartz spaces, measures, etc.

$^2$Not all locally finite measures are tempered, as illustrated by the example of $e^{t^2}dt$. Moreover, by signed measure, I mean a map $m$ satisfying the usual requirements of measures, except that it can take values in $(-\infty, +\infty]$ — it cannot take the value $-\infty$.

$^3$It is actually the point of view adopted in Daley and Vere-Jones [5], chapter 8.
This measure basically describes the behaviour of $\Phi$ around its typical point. In the litterature, it is often encountered under the notation $q^2$ or $g_2$.

**Proof of the theorem.** An easy application of Campbell’s formula (1.3) to the reduced Palm density $\mu_o$ will quickly show that the second term in the full covariance measure $\mu_{cov}$ (see the RHS of (1.5)) is actually given by

$$\lambda \int f(x)g(x + y)\mu_o(dy)dx = \int f \circ g(y)\lambda\mu_o(dy).$$

The first term is exactly equal to $\lambda f \circ g(0) = \int f \circ (x)\lambda\delta_0(dx)$, and the last one is obviously equal to $\int \int f(x)g(y)\lambda^2dxdy = \int f \circ g(x)\lambda^2dx$. Adding those terms yields the sought formula.

## 2.2 Scaled intersections

Apply the identity (2.2) to $f = g = \mathbf{1}_D$ for some measurable bounded set $D$ (typically, $D$ is the ball of radius $R$ around the origin): one obtains the formula

$$\text{Var}(\Phi \cap D) = \int_{\mathbb{R}^d} \alpha(x, D)\mathcal{C}(dx)$$

(2.5)

where $\alpha$ is the intersection volume; formally, $\alpha(x, D) = \mathbf{1}_D \circ \mathbf{1}_D(x) = |D \cap (x + D)|$ is the volume of the intersection between $D$ and $x + D$, a quantity which can efficiently be computed when $D$ is a simple shape such as an ellipsoid. The seminal paper [29] contains many examples of such computations.

We often like normalizing this expression with the volume of $D$ and developing the measure $\mathcal{C}$; formula (2.5) will often be encountered in the litterature as

$$\frac{\text{Var}(\Phi \cap D)}{|D|} = \lambda \left( \int_{\mathbb{R}^d} \alpha_2(x, D)\mathcal{C}(dx) \right)$$

(2.6)

where $\alpha_2(x, D) := \alpha(x, D)/|D|$ denotes the scaled intersection volume.

## 2.3 Structure factor and the phase-space variance formula

Shifting from the state space to the Fourier space will prove extremely useful for studying hyperuniformity. We recall (see Appendix (A.1)) that $\mathcal{F}$ is the Fourier transform of a tempered distribution.

**Definition 2.2.** The structure measure of a stationary point process $\Phi$ is the positive tempered measure defined by

$$S = 1 + \lambda \mathcal{F} \left[ \frac{\mu_o}{\lambda} - 1 \right].$$

(2.7)

It will also be refered to as structure factor or sometimes structure function.

The fact that $S$ is a positive measure actually follows from the following theorem, the second statement of which is also known as the phase-space variance formula.

**Theorem 2.3** (fundamental formulas). Let $f, g \in \mathcal{S}(\mathbb{R}^d)$ be Schwartz functions. Let $\Phi$ be a stationary random measure on $\mathbb{R}^d$ with intensity $\lambda$ and structure factor $S$. Then,

$$\text{Cov} \left( \sum_{x \in \Phi} f(x), \sum_{x \in \Phi} g(x) \right) = \frac{\lambda}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{f}(\xi)\hat{g}(\xi)S(d\xi)$$

(2.8)

and

$$\text{Var} \left( \sum_{x \in \Phi} f(x) \right) = \frac{\lambda}{(2\pi)^d} \int_{\mathbb{R}^d} |\hat{f}(\xi)|^2S(d\xi).$$

(2.9)

**Proof.** The Fourier transform of $\mathcal{C}$ is easily checked to be $\mathcal{F}\mathcal{C} = -(2\pi)^d\lambda^2\delta_0 + \lambda \mathbf{1} + \lambda \hat{\mu_o}$. Combining this with (2.2) and the reciprocal Fourier relation (A.10) yields the result.
Of course, equations (2.8)-(2.8) can be extended to wider classes of functions than Schwartz functions by appropriate limiting procedures. These equations notably hold true for indicator of bounded domains such as $B(0, r)$.

**Remark 2.4 (the structure kernel).** Any measurable function $f$ such that $|\hat{f}|^2 \in \ker(S)$ has zero variance, hence is deterministic: $\mathbb{P}$-almost surely one has

$$\sum_{x \in \Phi} f(x) = \lambda \int_{\mathbb{R}^d} f(x) dx.$$ 

This fundamental property will be exploited many times ahead, and explains why the structure of the kernel of $S$ is of paramount importance in the study of rigidities of point processes.

### 2.4 Number variance for simple shapes: Bessel and Sine kernels

In view of using the phase-space formula (2.8) applied to $f = 1_D$ with $D$ some window, it will prove useful to gather here a few Fourier transforms of classical shapes such as balls and boxes.

#### Balls and Bessel

Let $B = B(0, r)$ be the euclidean ball of radius $r$. In order to express the number variance as in (0) we recall the Fourier transform of $1_B$:

$$\hat{1_B}(\xi) = r^{d/2} \frac{J_{d/2}(r|\xi|)}{|\xi|^{d/2}}. \quad (2.9)$$

We also recall that the volume of $B(0, r)$ in a $d$-dimensional real space is $r^d$ times the volume of the unit ball $B(0, 1)$, given by

$$\kappa_d := \frac{\pi^{d/2}}{\Gamma(d/2 + 1)}. \quad (2.10)$$

If $\Phi$ is a stationary point process with intensity $\lambda$, formula (8) and formula (2.9) shows the following fundamental representation (see for instance eq.(58) in [28]):

$$\text{Var}(\Phi(B(0, r))) \frac{|B(0, r)|}{|B(0, r)|} = \lambda \int_{\mathbb{R}^d} J_{d/2}(r|\xi|)^2 S(d\xi). \quad (2.11)$$

Equivalently, we introduce $\mathcal{J}(x) = (\kappa_d(2\pi)^d)^{-1} r^{d/2} J_{d/2}(r|x|)^2 / |x|^d$. This is a smooth function except at zero, and it is nonnegative. Parseval’s identity ensures that $\int \mathcal{J}(x) dx = 1$. The family of functions $\mathcal{J}_\varepsilon(x) := \varepsilon^{-d} \mathcal{J}(x/\varepsilon)$ thus forms an approximation of unity \(^4\), which I will call the **Bessel approximations**, and we have

$$\text{Var}(\Phi \cap B(0, r)) \frac{|B(0, r)|}{|B(0, r)|} = \lambda \int_{\mathbb{R}^d} \mathcal{J}_{1/\varepsilon}(\xi) S(d\xi) \quad (2.12)$$

which goes to ‘the value of $S$ at zero’ under some assumptions on $S$ which will be described later.

#### Boxes and Sines

Let $C = B_\infty(0, r) = [-r, r]^d$ be the $\ell_\infty$-ball of radius $r$. We recall the Fourier transform of $1_C$:

$$\hat{1_{B_\infty(0, r)}}(\xi) = 2^d d^{d/2} \prod_{i=1}^d \sin(r \xi_i) \xi_i. \quad (2.13)$$

If $\Phi$ is a stationary point process with intensity $\lambda$, formula (8) and formula (2.13) show the following fundamental representation:

$$\text{Var}(\Phi(B_\infty(0, r))) \frac{|B_\infty(0, r)|}{|B_\infty(0, r)|} = \lambda \int_{\mathbb{R}^d} d\xi \prod_{i=1}^d \sin(r \xi_i)^2 S(d\xi) \quad (2.14)$$

where we used $|B_\infty(0, r)| = (2r)^d$.

\(^4\)But its support is not bounded.
3 Hyperuniformity and density fluctuations

The initial, intuitive and natural definition for hyperuniformity happens to be (0):
\[ \lim_{r \to \infty} \frac{\text{Var}(\Phi \cap B(0,r))}{|B(0,r)|} = 0. \]

However, there is a choice hidden in this definition: why choose euclidean balls? Is the definition equivalent if we choose, say, balls in the $\ell^p$ topology, or windows with different shapes such as ovals or rectangles?

This section clarifies the definition, introduces the phase-space characterization of hyperuniformity and the basic definitions linked with it: order metrics, hyperuniforms systems classification, quadratic variance formula.

3.1 On window shape dependence

Let us take a glimpse at the ‘lattice point counting problem’, in which we count lattice points in windows $D$ with different shapes; such a problem has extensively been studied during the past centuries and there are now powerful tools in these topics, but we will keep the exposition at a very elementary level.

It is easily understood that when $\Phi$ is a stationarized grid, ie $\Phi = U + \tau \mathbb{Z}^d$ with $U$ uniform on a unit cube and $\tau = 2\pi$, then the number of points in a centered square with side length $\ell$ has the same distribution than the number of points of $\mathbb{Z}^d$ in the square $U + [0, \ell]^d$ where $U$ is uniform over the unit cube. Let $D_{\ell}$ be this shifted square with side length $\ell > 0$. For simplicity we will only consider the dimension $d = 2$. Let $(u,v)$ be a uniform random variable on the square $[0,1] \times [0,1]$. The, the number of points of $\mathbb{Z}^2$ inside $D_{\ell} + (u,v)$ is the random variable given by

\[ N(\ell) = \lfloor \ell + u \rfloor \lfloor \ell + v \rfloor. \]

When $\ell$ is an integer this is simply $\ell^2$. We compute its second moment. Let us note $L = \lceil \ell \rceil$ and $r = L - \ell \in [0,1]$. Clearly, if $u \leq r$ then $\lfloor \ell + u \rfloor = L - 1$ and if $u > r$ then it is equal to $L$. Let us note $A = \{u, v \leq r\}, B = \{u, v > r\}$ and $C = \{u \leq r, v > r\} \cup \{u > r, v \leq r\}$, so that it is clear now that

\[ N(\ell) = (L - 1)^2 1_A + L^2 1_B + L(L - 1) 1_C \]

and as those events are disjoint, we get

\[ \mathbb{E}[N(\ell)^2] = \mathbb{P}(A)(L - 1)^4 + \mathbb{P}(B)L^4 + \mathbb{P}(C)L^2(L - 1)^2 \]
\[ = r^2(L - 1)^4 + (1 - r)^2L^4 + 2r(1 - r)L^2(L - 1)^2 \]
\[ = (\ell - [\ell])^2[\ell]^4 + ([\ell] - \ell)^2[\ell]^4 + 2(\ell - [\ell])[\ell] - [\ell]^2[\ell]^2. \]

If we take $\ell$ to be $n + 0.5$ for some integer $n$, then this simplifies and we get

\[ \sigma_{\ell} = \frac{\mathbb{E}[N(n + 0.5)^2] - (n + 0.5)^4}{(n + 0.5)^2} \]
\[ = \frac{0.5^2n^4 + 0.5^2(n + 1)^4 + 2 \times 0.5^2 \times n^2(n + 1)^2 - (2n + 1)^4 0.5^4}{(2n + 1)^2 0.5^2} \]
\[ = \frac{n^4 + (n + 1)^4 + 2n^2(n + 1)^2 - (2n + 1)^4 0.5^2}{(2n + 1)^2} \]
\[ = \frac{n^4 + n^4 + 4n^3 + 6n^2 + 4n + 1 + 2n^4 + 4n^3 + 2n^2 - 4n^4 - 8n^3 - 6n^2 - 2n - 1/4}{(2n + 1)^2} \]
\[ = \frac{n^2 + n + 3/8}{2n^2 + 2n + 1/2} \]

so that $\lim_{n \to \infty} \sigma_{n + 0.5} = 1/2$ which is notoriously nonzero; but this might seem counter-intuitive, since the window $D_{n + 0.5}$ is itself a totally honest convex ball with self-similar growth. Actually, this example can be pushed slightly further: we can tilt the squared window by some angle $\theta$. When $\tan(\theta)$ is a rational number, a resonance between the Fourier transform of the indicator of the tilted square and the Fourier transform of the lattice happens, as beautifully described in [18], causing nontrivial fluctuations.

The dependence on (0) with respect to window shape hopefully vanishes under some general hypotheses discussed in the following section.
3.2 Integrability of the covariance measure

If a point process satisfies (0), then by taking $D_n = B(0, n)$ in the scaled-intersection formula (2.6), one has

$$0 = \lim_{n \to \infty} \lambda \int \alpha_2(x, D_n) C(dv).$$

The smooth functions $x \mapsto \alpha_2(x, D_n)$ converge pointwise towards the constant function 1, and we would merrily pass the limit inside the former integral and conclude that

$$\int_{\mathbb{R}^d} 1 C(dv) = C(\mathbb{R}^d) = 0. \quad (3.1)$$

This identity is often seen in papers, but it really carries difficulties within. When $C$ is a difference between two measures, one cannot pass the limit inside the integral and this is not just a technical detail: the preceding section shows that one can find totally honest windows shapes $D_n$ for which the limit is equal to 0, and some others for which the limit is nonzero. If one wants to rigorously use (3.1), one has two choices:

1. either choose a specific limiting sequence $D_n$ growing to $\mathbb{R}^d$,
2. or make some hypothesis on $C$ which will legally allow to write (3.1).

I will choose the second road, but most of the incoming results might also be proved in a more general way using the first one. A simple and versatile hypothesis is the following.

**Hypothesis 3.1.** The covariance measure $C$ is a signed measure.

By Hahn’s theorem, this means that there are two measures $C_+, C_-$, the second one finite, such that $C = C_+ - C_-$. This happens for instance when $C$ has a bounded density $c$ with respect to the Lebesgue measure, and $c$ only takes negative values in a compact set. This will in general be the case for some determinantal point processes. The most blatant limitation of this hypothesis (but also, the only one I know) is that the class of lattices and perturbed lattices does not satisfy (3.1) — but they will have their own section in this survey.

When $C$ satisfies Hypothesis (3.1), one can pass to the limit inside both $C_+$ and $C_-$, and in this case, for any growing window shape $D_n$,

$$\lim_{n \to \infty} \frac{\text{Var}(|\Phi \cap D_n|)}{|D_n|} = \lim_{n \to \infty} \int \alpha_2(x, D_n) C_+(dv) - \int \alpha_2(x, D_n) C_-(dv)$$

$$= C_+(\mathbb{R}^d) - C_-(\mathbb{R}^d) \in (-\infty, +\infty].$$

In this case, (0) implies that $C(\mathbb{R}^d) = 0$ whatever the shapes for $D_n$, provided they are nested and grow to $\mathbb{R}^d$.

Note that when Hypothesis (3.1) is fulfilled, then the Fourier transform of $C$ is actually absolutely continuous with some continuous density that will be noted $\delta$, and furthermore one has

$$|\delta(\xi)| = \left| \int e^{-i(x, \xi)} C(dx) \right| \leq |C|(\mathbb{R}^d) < \infty. \quad (3.2)$$

Consequently, under (3.1), the structure measure assumes a particularly pleasant form: it is a bounded continuous function. We gather all the results in this section in the following.

**Proposition 3.2.** Let $\Phi$ be a stationary point process on $\mathbb{R}^d$, satisfying Hypothesis (3.1). Then, the following are equivalent:

1. $S(0) = 0$.
2. $C(\mathbb{R}^d) = 0$.
3. $\Phi$ satisfies (0).
4. $\Phi$ satisfies (0) for any sequence of nested, connected, convex open sets $D_n$ such that $D_n \nearrow \mathbb{R}^d$. 
3.3 General hyperuniformity

For any integrable function $f$, we define the normalized linear statistics variance of $\Phi$ at $f$ by

$$\sigma(f) := \frac{\text{Var} \left( \sum_{x \in \Phi} f(x) \right)}{\int_{\mathbb{R}^d} f(x)dx}. \quad (3.3)$$

**Definition 3.3** (hyperuniform statistics). Let $f$ be a measurable function. A stationary point process $\Phi$ is said to be hyperuniform with respect to $f$ if there is a sequence of Schwartz functions $f_n \in \mathscr{S}(\mathbb{R}^d)$ such that

1. $f_n \to f$ pointwise,
2. $\sigma(f_n) \to 0$.

We will often say that $f$ is a *hyperuniform statistic* for $\Phi$, and the set of hyperuniform statistics is going to be noted $\text{Hyp}(\Phi)$. The sequence $(f_n)$ will be called an approximation sequence. Usually, we want the constant function 1 to be a hyperuniform statistic for $\Phi$, which corresponds to the classical notion of number-hyperuniformity. Of course, my definition does not encompass the preliminary definition (0), but one can easily prove that (0) implies (1)-(2) above, by taking $f_n$ to be smooth functions taking the value 1 on $B(0,n)$ and zero outside $B(0,n+1)$.

3.4 Quadratic variance formula

**Theorem 3.4.** Let $\Phi$ be a hyperuniform point process satisfying Hypothesis (3.1). Then,

$$\text{Var} \left( \sum_{x \in \Phi} g(x) \right) = -\lambda^2 \int_{\mathbb{R}^d \times \mathbb{R}^d} |g(u) - g(u + v)|^2 \mathcal{C}(dv)du \quad (g \in \mathscr{S}(\mathbb{R}^d)). \quad (3.4)$$

This formula is common for determinantal point processes, where it is at the heart of the wonderful paper [3]; it was later used in full generality by Ghosh and Lebowitz in their paper [9] (look at equation (4) therein) to extend the results in [3]. Those results will be shown later on at Subsection 6.2 page 15.

Note that the integrated function $(u,v) \mapsto |g(u) - g(u + v)|^2$ is not a Schwartz function, it is only a bounded function. Without any hypothesis on $\mathcal{C}$, the integral above might be ill defined.

**Proof of (3.4).** Develop the right hand side. It is equal to the sum of three terms:

$$-\lambda^2 \int |g(u)|^2 \mathcal{C}(dv)du - \lambda^2 \int |g(u + v)|^2 \mathcal{C}(dv)du + \lambda \int g(u)g(u + v)\mathcal{C}(dv)du.$$

The first two ones are zero because for hyperuniform point processes satisfying Hypothesis (3.1) we have seen in Proposition 3.2 that $\mathcal{C}(\mathbb{R}^d) = 0$. The last one is equal to the variance in the LHS of (3.4) thanks to the covariance formula (2.2).

3.5 Stealthy hyperuniform processes

For most point processes, we just saw that hyperuniformity is equivalent to “$S(0) = 0$”. We could go further, and ask for $S$ to be zero in a neighborhood of zero.

**Definition 3.5.** A stationary point process $\Phi$ is called *stealthy hyperuniform* (SH for short) if there is an open set $U$ containing the origin, such that for any $f \in \mathscr{S}(\mathbb{R}^d)$ which is supported inside $U$, then $\langle S, |f|^2 \rangle = 0$.

The typical example is the stationary grid $U + \mathbb{Z}^d$. The spectral gap of a SH process is the largest $r$ such that $S$ vanishes on $B(0,r)$. For instance the spectral gap of stationarized grids is 1.

Such processes were notably studied, from a physicist point of view, in [32] and [30], and more recently in [12]. In Subsection 7 we prove that they have some extreme rigidity properties.
3.6 Hyperuniform systems classification

Let $\Phi$ be a hyperuniform point process. If the variance of the points in large balls is negligible compared to the volume, what is the good scaling? Does the variance behave like the surface area of the window, ie

$$\text{Var}(|\Phi \cap D_n|) \approx \text{Area}(\partial D_n)?$$

What behaviours could we expect for hyperuniform processes?

Let us place ourselves under Hypothesis (3.1), so that the structure function $S$ is actually an absolutely continuous measure with bounded density $\delta$ with $\delta(0) = 0$. Thanks to the Fourier representation of the variance derived in (2.9),

$$\sigma(r) := \frac{\text{Var}(|\Phi \cap B(0,r)|)}{|B(0,r)|} = \lambda \int_{\mathbb{R}^d} \hat{J}_1(r/\xi) \delta(\xi) d\xi,$$

we can efficiently study the behaviour of $\sigma(r)$ as dictated by the behaviour of $\delta(\xi)$ around zero using to classical asymptotics of Bessel functions as given by Lemma A.1 in the appendix.

**Proposition 3.6.** Let $\Phi$ be a hyperuniform stationary point process on $\mathbb{R}^d$ satisfying (3.1). We suppose that there is a constant $c$ such that $|\delta(\xi)| \sim c|\xi|^\alpha$ for $\xi$ in a neighborhood of zero. Then, the following happens according to the value of $\alpha$:

1. If $\alpha \in (0,1)$, then $\sigma(r) = O(1/r^\alpha)$.
2. If $\alpha = 1$, then $\sigma(r) = O(\log(r)/r)$.
3. If $\alpha > 1$, then $\sigma(r) = O(1/r)$.

**Proof.** We suppose wlog that $\lambda = 1$. Thanks to (2.12), we have

$$\sigma(r) := \frac{\text{Var}(|\Phi \cap B(0,r)|)}{|B(0,r)|} = \int_{\mathbb{R}^d} \hat{J}_1(r/\xi) \delta(\xi) d\xi = \int_{\mathbb{R}^d} \delta(x/r) \hat{J}_1(x) dx = \int_{|x|<r} \delta(x/r) \hat{J}_1(x) dx + \int_{|x|\geq r} \delta(x/r) \hat{J}_1(x) dx =: a(r) + b(r).$$

We supposed $\delta$ to be bounded, and we can wlog suppose it is always smaller than 1, so that $b(r) \lesssim \int_{|x|>r} \hat{J}_1(x) dx$. This integral is itself $O(1/r)$ thanks to Lemma A.1 above. Moreover, $a(r) \lesssim cr^{-\alpha} \int_{|x|<r} |x|^\alpha \hat{J}_1(x) dx$, and this integral is also easy to study thanks to the asymptotics in Lemma A.1. When $\alpha = 1$, it is equivalent to some constant times $\log(r)$, hence the case II. When $\alpha < 1$ it is bounded, and when $\alpha > 1$ it is $O(r^{-\alpha} \times r^{\alpha-1}) = O(1/r)$.

**Remark 3.7.** One might wonder if a decay smaller than 1/r is possible for $\sigma(r)$. It would mean that the variance of $|\Phi \cap B(0,r)|$ grows smaller than the surface of $B(0,r)$ which is $cr^{d-1}$. But this is actually not possible. For example, if $\mathcal{C}$ satisfies Hypothesis (3.1) and is absolutely continuous, the Riemann-Lebesgue lemma says that $\delta(x) \to 1$ as $x \to \infty$, hence in the proof above one has $\sigma(r) \gtrsim \int_{|x|>r} \hat{J}_1(x) dx \approx 1/r$. The same argument is also true under less restrictive assumptions, but at the cost of more technicalities.

Hyperuniform processes are frequently classified as types I, II and III, according if $\sigma(r)$ grows as 1/r, as $\log(r)/r$ or as $r^{-\alpha}$ for some $\alpha < 1$: see the paragraph 5.3.2 in Torquato’s survey [28]. We refer to the list in [28], Table 1, for many examples.

3.7 Order metrics

For Poisson point processes, one has $\mu_o = \lambda \mathbf{1}$, hence the term $\lambda^{-1} \mu_o - \mathbf{1}$ is exactly equal to zero. To compare the degree of order/disorder in a point process, a popular quantity is the so-called $\tau$-order-metric.
Definition 3.8. If $\mu_\circ$ has a density $\varrho$ with respect to the Lebesgue measure, then the $\tau$ order metric (TOM) is defined as
\[
\tau := \int_{\mathbb{R}^d} \frac{\varrho(x)}{\lambda} - 1 \, dx = |\lambda^{-1} \varrho - 1|^2.
\] (3.5)

The bigger the order metric, the stronger the order (at all scales) in the point process, with $\tau = 0$ being characteristic of Poisson processes.

If $\tau$ is finite, then by Parseval’s relation the Fourier transform of the total correlation function is also an $L^2$-function, hence in this case
\[
\tau = \frac{1}{\lambda^2} \int_{\mathbb{R}^d} |S(\xi) - 1|^2 \, d\xi.
\]

Another metric is sometimes used, the hyperuniform order metric (HOM). It is defined only for ‘class I HU processes’, i.e., processes with density fluctuations scaling as the window surface, i.e., they satisfy
\[
\frac{\text{Var}(|\Phi \cap B(0,r)|)}{r^{d-1}} \asymp 1.
\]

For such class-I HU processes, the HOM is defined as
\[
\Lambda = \lim_{r \to \infty} \frac{1}{r^d} \int_0^r \frac{\text{Var}(|\Phi \cap B(0,t)|)}{t^d} \, dt.
\] (3.6)

This is a Cesaro average of quantities bounded away from 0 and $\infty$, hence it cannot be zero nor $\infty$.

4 Finite sample analysis

Hyperuniformity, as defined earlier through (0), is a property of the probability distribution of $\Phi$; but how can we craft statistical procedures to detect hyperuniformity in one single realization of a point process, or even more realistically, in a finite-size approximation of one such realization?

4.1 Local weak convergence

Let $\mathbb{P}_n^\circ$ be a sequence of probability distributions on $\text{Conf}(\mathbb{R}^d)$, supported on the set $\text{Conf}_*(\mathbb{R})$ of configurations containing zero (such objects can be seen as ‘rooted’ point processes). For each $n$ we also note $\Phi_n$ a random variable with distribution $\mathbb{P}_n^\circ$.

Example 4.1 (Palm). We often choose $\mathbb{E}^\circ$ to be the Palm version $\mathbb{E}^\circ$ of a stationary point process on $\mathbb{R}^d$.

Example 4.2 (finite samples). Let $\varphi \in \text{Conf}(\mathbb{R}^d)$ be a nonempty finite set of points. It gives rise to a rooted random point process by simply randomly setting the origin of the space at one of the points of $\Phi$:
\[
\mathbb{P}_\varphi^\circ = \frac{1}{\text{card}(\Phi)} \sum_{x \in \Phi} \delta_{\varphi-x}.
\] (4.1)

When $\varphi$ is itself a realization of an almost surely finite random point process $\Phi$ with probability distribution $\mathbb{P}$, this definition leads to
\[
\mathbb{P}^\circ = \mathbb{E}[\mathbb{P}_\varphi^\circ]
\] (4.2)

which is an annealed probability measure.

We now define the Benjamini-Schramm convergence for geometric point processes.

Definition 4.3. We say that $\mathbb{P}_n^\circ$ converges locally weakly towards a (possibly random, possibly infinite) probability distribution $\mathbb{P}_\infty^\circ$ on $\text{Conf}_*(\mathbb{R}^d)$ if for any compactly supported smooth function $f$, one has
\[
\lim_{n \to \infty} \mathbb{E}_n^\circ \left[ \sum_{x \in \Phi_n} f(x) \right] = \mathbb{E}_\infty^\circ \left[ \sum_{x \in \Phi} f(x) \right].
\] (4.3)

We say that a sequence of finite point processes $\Phi_n$ converges towards a stationary point process $\Phi$ if the measures defined in Example converges in this sense towards the Palm version of $\Phi$. 

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This definition can be formulated in an equivalent setting: if \( \Phi_n \) has distribution \( P \circ \Phi_n \), then \( \Phi_n \) converges locally weakly to \( \Phi_\infty \) if for every \( r \),
\[
\Phi_n \cap B(0, r) \rightsquigarrow \Phi_\infty \cap B(0, r)
\]
where \( \rightsquigarrow \) denotes weak convergence of probability measures on \( \text{Conf}_* \).

**Proposition 4.4.** Let \( \Phi_n, n \in \bar{\mathbb{N}} \) be a sequence of rooted point processes with respective structure factors \( S_n \).
We suppose that \( \Phi_n \rightsquigarrow \Phi_\infty \). Then, \( S_n \rightsquigarrow S_\infty \) weakly.

We note \( \mu_{o,n}(A) = \mathbb{E}_n[|\Phi_n \cap A \setminus \{0\}|] \) the measure defined in (2.4). If \( A \) is relatively compact and if \( \Phi_n \) is lw-convergent to \( \Phi_\infty \), it is clear that \( \mu_{o,n} \rightsquigarrow \mu_{o,\infty} \).

### 4.2 Structure function for finite samples

Let \( \Phi \) be a finite set of points and let \( \mathbb{E}_o \) the probability distribution of \( \Phi - o \) where \( o \) is a uniformly chosen point in \( \Phi \). It is the exact analogue of the Palm distribution when \( \Phi \) is a stationary point process. Then measure \( \mu_o \) is then given as follows:
\[
\mu_o(A) = \mathbb{E}_o[(\Phi - o) \cap A]\]
\[
= \frac{1}{N} \sum_{x \in \Phi} |(\Phi - x) \cap A|
\]
\[
= \frac{1}{N} \sum_{x \in \Phi} \sum_{y \in \Phi \setminus \{x\}} 1_{y-x \in A}
\]
\[
= \frac{1}{N} \sum_{x \in \Phi} \left( \sum_{y \in \Phi} 1_{y-x \in A} - 1_{0 \in A} \right).
\]
So that
\[
\mu_o = -\delta_0 + \frac{1}{\text{card}(\Phi)} \sum_{x,y \in \Phi} \delta_{x-y}
\]
We recall that the structure factor is \( S = 1 + \mathcal{F}[\mu_o/\lambda - 1] \). One will quickly check the following useful identity.
\[
S(\xi) = \frac{1}{\text{card}(\Phi)} \left| \sum_{x \in \Phi} e^{-i\langle x, \xi \rangle} \right|^2.
\]
This is really easy to compute and to plot.

### 5 Some integrable examples

Program : DPP, Permanental, Pfaffian.

#### 5.1 Determinantal point processes

A determinantal point process (DPP) is a point process whose \( k \)-th correlation measure is absolutely continuous with respect to the Lebesgue measure\(^5\), and has density
\[
\rho^{(k)}(x_1, \ldots, x_k) = \det[(K(x_i, x_j))_{i,j \in [k]}]
\]
where \( K : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \) is a symmetric function called \( \text{kernel} \), which is supposed to be \( L^2 \). For general surveys on DPPs, we refer to [26] (a physical and statistical point of view) or to [16] for a complete mathematical investigation.

---

\(^5\)There are more general settings, but this one will be sufficient here.
Figure 1: Several realizations of planar point processes with around 1000 points, drawn with a colorplot representation of their structure factor (below). The color represents $S(\xi)$ for $\xi = (\xi_1, \xi_2)$ in a box of size 1000 around the origin. The darker, the closer to zero. The presence of a darker shape around zero indicates a hyperuniform point process ($S(\xi) \to 0$).

**Proposition 5.1.** A DPP with kernel $K$ is stationary if and only if
\[
\det([K(z + x_i, z + x_j)]_{i,j \in [k]}) = \det([K(x_i, x_j)]_{i,j \in [k]})
\]
for every integer $k$ and every $z, x_1, \ldots, x_k \in \mathbb{R}^d$.

This implies that $\lambda = K(x, x)$ for any $x$ and that there is a function $k$ such that $K(x, y) = k(x - y)$.

A basic consequence of the definition of correlation (1.4) in this setting is that the two-point correlation measure $\mu^{(2)}$ is a function: for any $f, g \in \mathcal{F}(\mathbb{R}^d)$, one has
\[
\mathbb{E} \left[ \sum_{x \neq y} f(x)g(y) \right] = \iint_{\mathbb{R}^d \times \mathbb{R}^d} f(x)g(y)g^{(2)}(x, y)dxdy
\]
where $g^{(2)}$ is the two-point correlation function:
\[
g^{(2)}(x, y) = \begin{vmatrix} K(x, x) & K(x, y) \\ K(y, x) & K(y, y) \end{vmatrix} = \lambda^2 - k(x - y)^2.
\]

This readily implies that $\mu_o = \lambda^{-1}[\lambda^2 - k(x)^2]dx$ and the covariance measure is given by $\mathcal{C} = \lambda \delta_0 - k(x)^2dx$. The structure function is then given by $1 - \lambda^{-1}[k^2]$. Upon noting $\mathfrak{H}(u) = \hat{k}(u)^2$ we thus see that $S$ is absolutely continuous with respect to the Lebesgue measure, and its density $s$ is
\[
s(\xi) = 1 - \frac{\mathfrak{H}(\xi)}{\lambda}.
\]

The value of the density function $s$ at zero is exactly $s(0) = 1 - \mathfrak{H}(0)/\lambda = 1 - \lambda^{-1} \int k(u)^2du$. The following proposition immediately follows.

**Proposition 5.2.** A stationary determinantal point process with $L^2$-kernel $K$ is hyperuniform if and only if
\[
K(0, 0) = \int_{\mathbb{R}^d} K(x, 0)^2dx.
\]

**Example 5.3.** For the Ginibre process, the correlation kernel is $K(x, y) = \pi^{-1}e^{-|x|^2/2-|y|^2/2}$ (where $x, y \in \mathbb{C}$). It is obviously stationary. Some Fourier transform computations show that in this case the Fourier transform has density
\[
s(\xi) = 1 - e^{-|\xi|^2/\lambda}.
\]

The Ginibre process is thus a class-I hyperuniform system. In dimension 2, it exactly gives the thermodynamic limit of the one-component plasma ([17]) at a specific temperature $1/\beta_0$. 

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6 Rigidity and hyperuniformity

**Definition 6.1.** Let $\Phi$ be a random point process on $\mathbb{R}^d$. A measurable function $f$ is a rigidity statistic for $\Phi$ if for any compact $K$, the random variable

$$\sum_{x \in \Phi \cap K} f(x)$$

is a measurable function of $\Phi \cap K^c$. The set of rigidity statistics of $\Phi$ will be noted $\text{Rig}(\Phi)$.

When $f \equiv 1$ is a rigidity statistic, we simply say that $\Phi$ is number-rigidity, which is the most common rigidity studied in the litterature. A maximal notion of rigidity is maximal rigidity, where we ask the whole configuration $\Phi \cap K$ to be a measurable function of $\Phi \cap K^c$. Such maximally rigid processes will be encountered in Subsection 7.

6.1 The variance argument

The basic tool for proving rigidity is an elegant Borel-Cantelli-type argument. It can be found in [10] and in many subsequent papers. We give a more general version than for number-rigidity.

**Proposition 6.2.** Let $\Phi$ be a point process. Suppose that there is a sequence of measurable functions $(f_n)$, such that $f_n \to f$ pointwise, and

$$\sum_{n=0}^{\infty} \text{Var} \left( \sum_{x \in \Phi} f_n(x) \right) < \infty. \quad (6.1)$$

Then $f$ is a rigidity statistic for $\Phi$.

**Proof.** Note $S_n = \sum f_n(x)$ and $e_n = E[S_n]$ and $\varepsilon_n = \text{Var}(S_n)$. Let $s_n$ be a sequence of numbers going to zero and such that $\sum \varepsilon_n/s_n < \infty$ (such a sequence always exists). Then by Chebyshev’s inequality,

$$\sum_{n=1}^{\infty} P(\left| S_n - e_n \right| > \sqrt{s_n}) \leq \sum_{n=1}^{\infty} \varepsilon_n/s_n < \infty.$$

By the Borel-Cantelli lemma, almost surely one has $|S_n - e_n| \leq \sqrt{s_n} = o(1)$ for any $n$ large enough. Define

$$R_n = \sum_{x \in \Phi \cap K^c} f_n(x)$$

which is a measurable function of $\Phi \cap K^c$. Then, by the properties of $f_n$, the following statements are almost surely true:

$$\sum_{x \in \Phi \cap K} f(x) = \lim_{n \to \infty} \sum_{x \in K^c \Phi} f_n(x) = \lim_{n \to \infty} S_n - R_n = \lim_{n \to \infty} e_n - R_n$$

and the latter is a limit of random variables which are $\Phi \cap K^c$-measurables, hence it is overall $\Phi \cap K^c$-measurable and $\Phi$ is $f$-rigid.

Thus, proving rigidity amounts to proving serious variance bounds. Note that one can also relax further this kind of criteria: the core argument of the proof is that there is some measurable function $f_n$ with $f_n \to f$ pointwise, and such that $\sum_{x \in \Phi} f_n(x)$ is $o(1)$-close to a $\Phi \cap K^c$-measurable random variable — in our case, its own expectation $e_n$.

**Remark 6.3 (rigidity criterion and hyperuniformity).** Note that if such functions $(f_n)$ do exist, then the variance $\text{Var}(\sum f_n(x))$ goes to zero. If the functions are chosen so that $\int f_n(x)dx$ is bounded away from zero, this immediately implies $\sigma(f_n) \to 0$, hence $f$ is a hyperuniform statistic for $\Phi$ in the sense of Definition 3.3. The assumption of the preceding proposition is thus extremely strong, showing both hyperuniformity and rigidity.
6.2 Rigidity from hyperuniformity: one and two dimensions

A crucial result obtained by Ghosh and Lebowitz is that in dimensions 1 and 2, hyperuniformity implies rigidity, as long as the correlations are sufficiently decaying ([9]). This is no longer true — and actually pretty mysterious — for higher dimensions, as discussed in the next section. By ‘sufficiently decaying’, we mean that the measure has four finite moments.

Hypothesis 6.4. A signed measure \( m \) has strong decay if its moments of order 1, 2, 3, 4 are finite.

Theorem 6.5 ([9]). Let \( \Phi \) be a stationary point process on \( \mathbb{R}^d \) with \( d = 1 \) or 2. If \( \mathcal{C} \) has strong decay and \( \Phi \) is hyperuniform, then \( \Phi \) is rigid.

Note that by the definition of \( \mathcal{C} \), the existence of moments for \( \mathcal{C} \) is equivalent with the existence of moments for the measure \( \mu_\circ - \lambda \mathbb{1} \).

Proof. We only do the proof for \( d = 2 \), the proof for \( d = 1 \) being essentially simpler. The goal is to prove rigidity using the variance criterion 6.1, and to construct a measurable function \( \Phi \) taking the value 1 on \( B(0, r) \), such that \( \text{Var}(\sum f(x)) < \varepsilon \) for arbitrary \( \varepsilon > 0 \).

We choose some \( r > 1 \). Let \( g \) be a smooth function supported on \( B(0, R) \), smaller than 1 everywhere, and taking the value 1 in \( B(0, 1) \). We set \( g_b(x) = g(x/b) \) and \( S_b = \sum_{x \in \Phi} g_b(x) \). We start with (3.4):

\[
\text{Var}(S_b) = \frac{-\lambda}{2} \iint |g_b(u) - g_b(u + v)|^2 \mathcal{C}(dv)du.
\]

First of all, note that if \( |u| > bR \) and \( |u + v| > bR \), the integrand is zero, hence we only have to integrate on the union of \( A_1 = \{|u| \leq bR \} \) and \( A_2 = \{|u + v| \leq bR \} \); by the triangle inequality, we thus have

\[
\text{Var}(S_b) \leq \frac{\lambda}{2} \iint_{A_1} |g_b(u) - g_b(u + v)|^2 \mathcal{C}(dv)du + \frac{\lambda}{2} \iint_{A_2} |g_b(u) - g_b(u + v)|^2 \mathcal{C}(dv)du.
\]

and we separately bound those terms. To do this, we write the Taylor development of \( g_b \) as

\[
g_b(u + v) = g_b(u) + b^{-1} \langle \nabla g(u/b), v \rangle + h(u, v)
\]

where \( |h(u, v)| \leq c_1|v|^2/b^2 \) for some constant \( c_1 \) depending on \( g \) (indeed, \( c_1 = \sup \|H^2 g(x)\| \) where \( H^2 g \) is the Hessian matrix). We have

\[
|g_b(u) - g_b(u + v)|^2 \leq \frac{\nabla g(u/b)^2|v|^2}{b^2} + \frac{2|\nabla g(u/b)||v||h(u, v)|}{b} + |h(u, v)|^2
\]

\[
\leq \frac{\nabla g(u/b)^2|v|^2}{b^2} + \frac{2|\nabla g(u/b)|c_1|v|^3}{b^3} + \frac{c_1|v|^4}{b^4}.
\]

Let \( c \) be a constant greater than first four moments of \( \mathcal{C} \), ie \( c \geq \int |x|^k |\mathcal{C}|(dx), k \in \{1, 2, 3, 4\} \). We have

\[
I_1 \leq \int_{B(0,bR)} \int \left( \frac{\nabla g(u/b)^2|v|^2}{b^2} + \frac{2|\nabla g(u/b)|c_1|v|^3}{b^3} + \frac{c_1|v|^4}{b^4} \right) |\mathcal{C}|(dv)du
\]

\[
\leq c \int |\nabla g(y)|^2dy + \frac{2cc_0\pi b^2 R^2 c_1}{b^3} + \frac{\pi b^2 R^2 c_1 c}{b^4}
\]

\[
\leq c \int |\nabla g(y)|^2dy + O\left(\frac{cc_0 c_1 R^2}{b}\right) + O\left(\frac{c_1 R^2}{b^2}\right)
\]

\[\text{We recall that } |\mathcal{C}| \text{ is the 'absolute value' of the signed measure } \mathcal{C}.\]
where we also introduced \( c_0 = \sup |\nabla g(x)| \), and the constants in the \( O(\cdot) \) is universal. A similar bound holds for \( I_2 \). Consequently,

\[
\Var(S_0) \leq c \int |\nabla g(y)|^2 \, dy + \frac{O(c_0 c_1 R^2)}{b} + \frac{O(c_1 R^2)}{b^2}\quad (6.2)
\]

The last step is thus to find, for any \( \varepsilon > 0 \), a function \( g \) as described in the beginning of the proof, and such that this last bound is smaller than \( \varepsilon \). This can be done, and it is the content of the lemma thereafter. We first choose \( g_{c/21c} \), so that the first term in the RHS of (6.2) is smaller than \( \varepsilon/3 \). Then, \( c_0, c_1 \) and \( R \) are fixed (they might depend on \( \varepsilon \)), and we take \( b \) so large that the two remaining terms in the RHS of (6.2) are smaller than \( \varepsilon/3 \).

**Lemma 6.6.** For every \( \varepsilon > 0 \), there is a function \( g_\varepsilon : \mathbb{R}^2 \to \mathbb{R} \) which as the following properties: it is nonnegative, takes the value 1 on \( B(0,1) \) and 0 outside \( B(0,R_\varepsilon) \) for some \( R_\varepsilon > 1 \), it is of class at least \( \mathcal{C}^2 \), it is smaller than 1 everywhere, and it satisfies

\[
\int |\nabla g_\varepsilon(x)|^2 \, dx < 7\varepsilon.\quad (6.3)
\]

We explain a construction which can be found in [13].

**Proof.** Let us start by setting \( R'_\varepsilon = e^{2/\varepsilon} \). We now a real-valued function \( a : [0,\infty[ \to [0,\infty] \) by

\[
a(t) = \begin{cases} 
1 & \text{if } t \in [0,1], \\
1 - \log(t)\varepsilon/2 & \text{if } t \in [1,R'_\varepsilon] \\
0 & \text{else}.
\end{cases}
\]

This function is continuous and it is smooth, except at 1 and \( R'_\varepsilon \). Moreover, for \( t \) between 1 and \( R'_\varepsilon \), we have \( a'(t) = -\varepsilon/2t \) and \( a''(t) = \varepsilon/2t^2 \). These properties ensure that we can find a nonnegative \( \mathcal{C}^2 \) function \( b \) with \(|b'(t)| \leq \varepsilon/t, |b''(t)| \leq \varepsilon/t^2 \), taking the value 1 on \( B(0,1) \), smaller than 1 everywhere, and being zero outside \( B(0,R_\varepsilon) \) with \( R_\varepsilon = 1 + e^{2/\varepsilon} \).

We simply define \( g_\varepsilon(x) = b(|x|) \). We have \( \nabla g(x) = b'(|x|)x/2|x| \), hence

\[
|\nabla g|^2 = 2\pi \int_1^{R_\varepsilon} |\nabla g(r)|^2 r dr d\theta \leq 2\pi \int_1^{R_\varepsilon} \frac{b'(r)^2 r^3}{2r^2} \, dr \leq 2\pi \int_1^{R_\varepsilon} \frac{\varepsilon^2 r^2}{2r} \, dr \leq \pi \varepsilon^2 \log(1+R_\varepsilon) \leq 7\varepsilon.
\]

This settles the lemma. \( \square \)

**Remark 6.7 (d \geq 3).** The crucial fact in the preceding proof is that we can find \( g_b \) such that (6.2) is true, and in particular the first term does not depend on \( b \). In dimensions higher than 3, there would be a \( b \) factor there and the subsequent argument would fail.

### 6.3 Rigidity from hyperuniformity: three dimensions and more

Consider the perturbed lattice: start with the stationarized lattice \( U + \mathbb{Z}^d \), then replace each point \( x \) by \( x + H_x \) where \( (H_x) \) is a collection of iid random variables, say centered Gaussian variables with variance \( \sigma^2 \). The corresponding process will be \( \Phi_\sigma \). It turns out that there is a surprising threshold for number rigidity, as proved by Sly and Peres in their 2014 paper [25].

**Theorem 6.8** (Peres and Sly). If \( d \geq 3 \), there is a \( \sigma_c(d) > 0 \) such that \( \Phi_\sigma \) is number-rigid if \( \sigma < \sigma_c \) and not rigid if \( \sigma > \sigma_c \).

But one can also prove that such processes always satisfy (0) and have very strong decay of correlations. The strict analogue of the results in Subsection 6.2 cannot hold, for those processes with \( \sigma > \sigma_c \) are hyperuniform and not rigid. Ghosh and Lebowitz conjecture that the relation should be the other way round, namely that rigidity should imply hyperuniformity. They argue that (verbatim from [12]) “when the variance grows like |\( \Lambda \) [the window], it behaves in an additive way for two adjacent domains which seems to suggest that surface effects become negligible for large |\( \Lambda \), which is inconsistent with number rigidity”. 

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7 Extreme rigidity for stealthy hyperuniform processes

This section demonstrates that SH processes are extremely rigid: points cannot accumulate too much but they also cannot leave big holes, and the whole process is maximally rigid in the sense defined right after Definition 6.1 at page 14. These beautiful results entirely rely upon the Fourier approach to hyperuniformity; everything in this section comes from [12]. In the proofs I mention a few ‘facts’ which are actually purely analytic and independent of our problems. Their justification is postponed to the end of this section.

7.1 Anticoncentration and bounded holes

Theorem 7.1 (anti-concentration). Let \( \Phi \) be a SH point process on \( \mathbb{R}^d \) with intensity \( \lambda \) and with spectral gap \( r \) or more. Then, there are two constants \( c_0, r_1 \) such that for any open set \( V \) with diameter smaller than \( c_0/r \), the number of points of \( \Phi \) in \( V \) is smaller than \( \lambda c_0 \).\( r_1^d \).

Proof. As usual, the key is given by the phase-space variance formula (\( \ast \)). If \( f \) is any function such that \( f \) is supported inside \( B(0, r) \), then \( \langle S, \hat{f}^2 \rangle = 0 \), hence the variance of \( S(f) \) is zero and this random variable is actually almost surely equal to its expectation \( \lambda \int f(x)dx \). Our goal is to show the following fact.

Fact 1. There is a (deterministic) function \( f \) and a constant \( c_0 \), and such that

1. \( f \) is nonnegative and \( f(x) \geq r^d \) for any \( |x| \leq c_0/r \),
2. \( f \) has support inside \( B(0, r) \).

If such things exist, then almost surely one has

\[
|r^d|\Phi \cap B(0, c_0/r)| \leq \sum_{x \in \Phi} f(x) = \lambda \int_{\mathbb{R}^d} f(x)dx =: c_1.
\]

One can then easily use the mass transport principle to show that this is indeed true for any \( B(x, c_0/r) \), and for any open set with diameter smaller than \( c_0/r \).

Theorem 7.2 (Bounded holes theorem). Let \( \Phi \) be a SH point process on \( \mathbb{R}^d \) with intensity \( \lambda \) and with spectral gap \( r \) or more. There is a constant \( c \) such that almost surely, any ball of radius \( c/r \) contains at least one point of \( \Phi \).

Proof. We will suppose that the intensity is 1. Set \( R > 0, \delta = c_0/r \) with \( c_0 \) the constant in the Anti-concentration Theorem, and define the square annulus \( B_\ell = \{ \delta(R + \ell) \leq |x| < \delta(R + \ell + 1) \} \) for every \( \ell > 0 \). For \( \ell = 0 \) we set \( B_0 = B_\infty(0, \delta R) \).

Let \( f \) be the function whose existence was proved in Fact 1, so that almost surely one has

\[
\sum_{\ell=0}^{\infty} \sum_{x \in \Phi \cap B_\ell} f(x) = c_1 > 0.
\]

Define \( X_\ell \) to be the \( \ell \)-th term in this sum. The sketch of the proof is as follows: one can show that for \( \ell > 0 \), the random variable \( X_\ell \) is \( O((R + \ell)^{-2}) \). Hence, if \( R \) is chosen so that there are no points in \( B_0 \), then the whole sum above is going to be \( \sum_{n \geq R} n^{-2} = O(R^{-1}) \), which shows that \( R \) cannot be too big.

Formally, let \( E = E(R) \) be the event \{ \( \Phi \) has no points in \( B_0 \) \}. If this event had zero probability for every \( R \), this would mean that \( \Phi \) has an atom at zero, which is not true since \( \Phi \) is stationary. We thus choose some \( R > 0 \) such that \( \Pr(\overline{E(R)}) > 0 \) and from now on we will work on this event: there are no points of \( \Phi \) in \( B_0 \).

The sum in (7.1) can thus be started at \( \ell = 1 \).

Fact 2. Almost surely,

\[
\forall \ell > 0, \quad X_\ell \leq \frac{c_2}{r} \frac{|\Phi \cap B_\ell|}{|(R + \ell)\delta|^{d+1}}
\]

for some constant \( c_2 \).

\footnote{And I will probably write them later this year.}
As a byproduct, we also have the bound satisfies \( \hat{\zeta} \)

Using this bound and Fact 2 to bound every term in the LHS of (7.1), one gets Fact 3.

Actually, one can take fewer than \( \delta \) square boxes and with diameter \( \delta \).

\[
\sum_{\ell=1}^{\infty} \frac{1}{(R+\ell)^2} \leq \frac{2c_1c_2c_3}{c_0^2} \frac{R}{R^d-1}.
\]

As a byproduct, we also have the bound \( R \leq 2c_2c_3/c_0^2d-1 \).

### 7.2 Maximal rigidity of SH processes

We recall that a PP is maximally rigid if \( \Phi \cap K \) is a measurable function of \( \Phi \cap K^c \): if the points outside the compact \( K \) are revealed, then we are able to entirely reconstruct the points of \( \Phi \) in \( K \). The proof is adapted from [12].

**Theorem 7.3** (maximal rigidity). Stealthy hyperuniform point processes are maximally rigid.

**Proof.** It is enough to prove such a theorem when \( K \) is a ball and for simplicity we assume that \( \lambda = 1 \). The set of points \( \Phi \cap K \) is as finite, and is entirely characterized by the Fourier transform

\[
\chi(\xi) := \sum_{x \in \Phi \cap K} e^{-i(\xi,x)}
\]

hence we only have to prove that \( \chi \) a measurable function of \( \Phi \cap K^c \). Consider a small ball \( B(0,3\delta_0) \) entirely included in \( U \), the open set on which the structure measure \( \mu \) vanishes. Let \( \delta < \delta_0 \). Then the function \( h = 1_{B(0,\delta)} \ast 1_{B(0,\delta)} \) is supported inside \( B(0,2\delta) \subset U \). Using (2.9) we see that the function defined by

\[
f_{\delta}(x) = \frac{\delta^d J_{d/2}(\delta|x|)^2}{|x|^d}
\]

satisfies \( \hat{f}_{\delta} = h_\delta \). We now set \( F_{\mu,\delta}(x) := e^{i(x,\mu)} f_{\delta}(x) \). Thanks to the symmetries of the Fourier transform, this function satisfies \( \hat{F}_{\mu,\delta}(\xi) = h_\delta(\xi - \mu) \), hence if \( |\mu| < \delta_0 \) it is supported inside \( B(\mu,2\delta) \subset B(0,3\delta_0) \subset U \); it is in the kernel of \( \delta \), hence the random variable \( \sum_{x \in \Phi} F_{\mu,\delta}(x) \) has zero variance (see Remark 2.4) and one has almost surely

\[
\sum_{x \in \Phi \cap K} e^{i(x,\mu)} f_{\delta}(x) = \sum_{x \in \Phi \cap K} F_{\mu,\delta}(x) = c(\mu,\delta) - \sum_{x \in \Phi \cap K^c} F_{\mu,\delta}(x) \tag{7.3}
\]

where \( c(\mu) = \int F_{\mu,\delta} \). The RHS is a measurable function of \( \Phi \cap K^c \), say \( G_{\delta}(\Phi \cap K^c,\mu) \). Moreover, (7.3) is almost surely simultaneously true for all \( \mu \in B(0,\delta_0) \) with rational coordinates and \( \delta \) rational and small. Note that thanks to the Bessel asymptotics \( J_{\nu}(x) \sim \Gamma(\nu+1)^{-1}(x/2)^{\nu} \) when \( x \to 0 \), we see that \( f_{\delta}(x)/\delta^{2d} \to C(d) > 0 \) when \( \delta \to 0 \), where \( C(d) \) is a constant. We divide each member of (7.3) by \( C(d)\delta^{2d} \) and take the limit along a sequence of rational numbers (\( \delta_n \)) going to zero.

\[
\chi(\mu) = \sum_{x \in \Phi \cap K} e^{i(x,\mu)} = \lim_{n \to \infty} \sum_{x \in \Phi \cap K} e^{i(x,\mu)} = \lim_{n \to \infty} \frac{G_{\delta_n}(\Phi \cap K^c,\mu)}{C(d)\delta_n^{2d}}.
\]

This is true for all \( \mu \in B(0,\delta_0) \) with rational coordinates. But \( \chi \) is continuous, hence it is also a measurable function of \( \Phi \cap K^c \) as requested. \( \Box \)

**Question 7.4.** Let \( \Phi \) be a SH point process and \( \Psi \) an infinite subset of its points. Is \( \Phi \) a measurable function of \( \Psi \)? In other words, if I remove an infinite number of points of \( \Phi \), can I recover \( \Phi \)?

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8 The Coulomb gas

We review in this section some questions about rigidities in three-dimensional (and higher) Coulomb systems, and a major result obtained in three dimensions by Sourav Chatterjee in 2017 in his paper [4], then recently extended to all dimensions by Ganguly and Sarkar in [8]: hyperuniformity for the hierarchical Coulomb gas.

8.1 Definition of the Hierarchical Coulomb gas

The Coulomb gas in \(d\) dimensions is a random set of \(n\) points in \([0,1]^d\) whose density with respect to the Lebesgue measure is

\[
\frac{1}{Z(\beta)} \exp \left\{ -\beta \sum_{i \neq j} w(x_i, x_j) \right\}.
\]

(8.1)

Here \(Z(\beta)\) is the partition function, \(\beta\) is the inverse temperature and \(w\) is an interaction. In the case of the hierarchical Coulomb gas, this interaction is defined as follows. We divide the unit cube \([0,1]^d\) into \(2^d\) small cubes of side length \(1/2\), then we divide each of these cubes, etc. For \(x \neq y\) in the unit cube, we set

\[
w(x,y) = 2^{(d-2)(k-1)}
\]

(8.2)

where \(k\) is the first integer so that \(x\) and \(y\) do not belong to the same cube with side length \(2^{-k}\).

The point process thus obtained will be noted \(\Phi_n\).

8.2 Hyperuniformity and rigidity of Coulomb systems

In [24], Coulomb systems in 3 dimensions were non-rigorously proved to be hyperuniform.

8.3 Hyperuniformity of the HCG

The dimension \(d\) is always supposed to be 3 or more. Suppose that \(U\) is an open set in the unit cube. Then the number of points inside \(U\) should be proportional to \(n\), hence we would like to prove something like \(\text{Var}(|\Phi_n \cap U|)/n = o(1)\). This will be the content of the theorem, and for technical reasons we restrict to admissible open sets, which are :

- nonempty, open, connected,
- whose boundary is a smooth closed orientable surface.

Note that this encompasses virtually all the window shapes, not only balls.

**Theorem 8.1** (Macroscopic hyperuniformity, [4, 8]). Let \(U\) be an admissible open subset of \([0,1]^d\). Then \(\mathbb{E}[|\Phi \cap U|] = |U|n\). Moreover, there are two positive constants \(c(U, \beta), C(U, \beta)\) such that

\[
\frac{c(U, \beta)}{n^{1/d}} \leq \frac{\text{Var}(|\Phi_n \cap U|)}{n} \leq \frac{C(U, \beta) \log(n)^{13}}{n^{1/d}}
\]

(8.3)

9 Lattices and perturbed lattices

Gabrielli : [7]. Recovering the lattice : [31]. Sodin and Tsirelson : [27]. Rigidity and tolerance : [25]. Cloaking : [19]. Matchings : [20].
Appendix A  Fourier analysis

A.1 Tempered distributions and the Schwartz space

We list here the few basic results we will need on tempered distributions and their Fourier transforms; they can be found in any introductory book to real analysis. We refer to Laurent Schwartz’s Cours d’analyse or to [15].

A function $f : \mathbb{R}^d \rightarrow \mathbb{C}$ is a Schwartz function if it has derivatives at every order and if for any multi-index $\alpha, \beta$ one has

$$p_{\alpha, \beta}(f) := \sup_{x \in \mathbb{R}^d} |x^\beta \partial_\alpha f(x)| < \infty.$$  

The maps $p_{\alpha, \beta}$ are semi-norms and they generate the topology of the set $\mathcal{S}(\mathbb{R}^d)$ of every Schwartz function.

A continuous linear form over $\mathcal{S}(\mathbb{R}^d)$ is called a tempered distribution and the set of tempered distributions is noted $\mathcal{S}'(\mathbb{R}^d)$. The Fourier transform of a Schwartz function $f$ is the Schwartz function defined by

$$\mathcal{F}(f)(\xi) := \int_{\mathbb{R}^d} f(x) e^{-i\langle \xi, x \rangle} dx$$

and the inverse Fourier transform is $\mathcal{F}^{-1}(f)(\xi) := \mathcal{F}(f)(-\xi)$. The Fourier transform is an isomorphism of $\mathcal{S}(\mathbb{R}^d)$ and one has the fundamental reciprocity relation

$$\mathcal{F} \mathcal{F} = (2\pi)^d I.$$  

(A.1)

The Fourier transform of a tempered distribution $S \in \mathcal{S}'(\mathbb{R}^d)$ is then defined through duality by

$$\langle \mathcal{F}S, f \rangle = \langle S, \mathcal{F}f \rangle.$$ 

This is an isomorphism on $\mathcal{S}'(\mathbb{R}^d)$ and (A.1) still holds.

If $g$ has sub-polynomial growth one can define the multiplication of (the tempered distribution associated with) $g$ and a tempered distribution $S$ through the formula

$$\langle g.S, f \rangle := \langle S, gf \rangle.$$ 

If $g$ is $L^1$, then one can also define the convolution of $g$ with a tempered distribution $S$ through the formula

$$\langle g * S, f \rangle := \langle S, g \ast f \rangle.$$  

(A.2)

With those definitions, the group morphism property of Fourier transforms is preserved:

$$\mathcal{F}(g * S) = \mathcal{F}g \mathcal{F}S.$$  

(A.3)

A.2 Special Fourier transforms

Let $B(0, r)$ be the euclidean ball of radius $r$ around the origin in $\mathbb{R}^d$. Its Fourier transform is given by the following formula

$$\hat{1}_{B(0, r)}(\xi) = r^d/2 \frac{J_d/2(r|\xi|)}{|\xi|^{d/2}}.$$  

(A.4)

where $J_\nu$ is the Bessel function of the first kind of index $\nu$. We recall that the basic definition of the Bessel function is

$$J_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+\nu+1)} \left(\frac{z}{2}\right)^{2n+\nu}.$$ 

Proof. Set $B = B(0, 1)$. As $1_B$ is rotation-invariant, so is its Fourier transform, hence it is enough to compute the Fourier transform at $\xi = (0, \ldots, 0, t)$. Let $\omega_k$ be the volume of the $k$-dimensional unit ball $^8$. Integration by slices yields

$$\hat{1}_B(\xi) = \int_B e^{-ix \cdot \xi} dx_1 \ldots dx_d = \omega_{d-1} \int_{-1}^1 e^{-ixt}(\sqrt{1-x^2})^{d-1} dx.$$ 

\footnote{One has $\omega_k = \frac{x^{k/2}}{\Gamma(1+k/2)}$.}
Now the Poisson representation formula for Bessel functions, valid for any \( \nu > -1/2 \), is
\[
J_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu + 1/2)} \int_{-1}^{1} e^{-izt} (1 - t^2)^{\nu-1/2} dt
\]  
(A.5)
where \( J_\nu \) is the Bessel function of the first kind of order \( \nu \). Setting \( \nu = d/2 \), we get
\[
\hat{1}_B(\xi) = \frac{\omega_d \sqrt{\pi} \Gamma((d+1)/2)}{(t/2)^{d/2}} J_{d/2}(t) = \frac{J_{d/2}(|\xi|)}{|\xi|^{d/2}}.
\]

By homogeneity\(^9\), we obtain (2.9)

We will need some asymptotics of Bessel functions. Such asymptotics can be easily found on any book on special functions, such as [22].

**Lemma A.1.** Let \( \nu \) be a positive real number. As \( x \to \infty \), one has
\[
J_\nu(t) = \sqrt{\frac{2}{\pi t}} \left( \cos(t - c_\nu) + O\left(\frac{1}{t}\right) \right)
\]  
(A.6)
where \( c_\nu := \nu \pi/2 + \pi/4 \). The positive real function \( j: x \in \mathbb{R}^d \mapsto J_{d/2}(|x|)/|x|^d \) thus satisfies the following bound as \( |x| \to \infty \):
\[
j(x) = O\left(\frac{1}{|x|^{d+1}}\right).
\]  
(A.7)

We now turn to boxes. We note \( B_\infty(0,r) \) the set of \( x \in \mathbb{R}^d \) with all coordinates between \(-r\) and \( r \), ie the box of side length \( 2r \). Its Fourier transform is given as follows.
\[
\hat{1}_{B_\infty(0,r)}(\xi) = 2^d \prod_{i=1}^{d} \frac{\sin(\xi_i)}{\xi_i}.
\]  
(A.8)

**Proof.** Set \( \xi = (\xi_1, \ldots, \xi_d) \). Then,
\[
\int_{B_\infty(0,1)} e^{-i(\xi,x)} \, dx = \int_{-1}^{1} \cdots \int_{-1}^{1} e^{-i(\xi_1,x_1)} \cdots e^{-i(\xi_d,x_d)} \, dx_1 \cdots dx_d
\]
\[
= \prod_{i=1}^{d} \left( \int_{-1}^{1} e^{-i\xi_i x} \, dx \right)
\]
\[
= 2^d \prod_{i=1}^{d} \frac{\sin(\xi_i)}{\xi_i}.
\]

Identity (2.13) follows from homogeneity. \(\square\)

### A.3 Positive, positive-definite (ppd) measures

A Borel measure \( m \) on \( \mathbb{R}^d \) is ppd if it is a positive Borel measure which is also positive definite, which means that for any compactly supported test function \( f \), one has
\[
\int_{\mathbb{R}^d} f \ast \hat{f}(x) m(dx) \geq 0
\]  
(A.9)
where \( \hat{f}(x) := f(-x) \). As we will often have to use the convolution \( f \ast \hat{g} \), I find it easier to use the notation \( f \otimes g \) instead. We refer the reader to the first chapter of Berg and Frost’ *Potential theory* book for a clear treatment of this theme ([11]).

\(^9\)If \( f(x) = g(x/r) \) then \( \hat{f}(\xi) = |r|^d \hat{g}(r \xi) \).
The set of pPD measures is noted $\mathcal{P}_{\text{pd}}$. A measure is *signed pPD* if it is the difference of two pPD measures. It turns out that pPD measures are the good framework for working with Fourier transform of measures which are not necessarily finite. We refer the reader to the first chapter of Berg and Frost’s *Potential theory* book for a clear treatment of this theme ([1]). It is not obvious how to define Fourier transforms of infinite measures in general, even if they are locally finite; for instance, the measure $e^{-x^2}dx$ is locally finite but it does not define a tempered distribution, hence there is no obvious way to define its Fourier transform. However, pPD measures behave well.

**Proposition A.2.** $\mathcal{P}_{\text{pd}} \subset \mathcal{S}'(\mathbb{R}^d)$.

Consequently, for any signed pPD $m$, one can define $\mathcal{F}m$, which is a tempered distribution; but Bochner’s theorem comes into play and guarantees that $\mathcal{F}m$ is indeed a signed measure.

**Theorem A.3 (Ultimate Bochner Theorem).** *The tempered Fourier transform is an isomorphism of $\mathcal{P}_{\text{pd}}$ and the inversion formula holds.*

In other words, if $m$ is pPD and has Fourier transform $\hat{m}$, then for any $f \in \mathcal{S}(\mathbb{R}^d)$ one has

$$
\int_{\mathbb{R}^d} f(x)m(dx) = (2\pi)^{-d} \int_{\mathbb{R}^d} \hat{f}(x)\hat{m}(dx)
$$

(A.10)

which also known as the *reciprocity relation*. 

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References


